EXOTIC BIALGEBRAS : NON-DEFORMATION QUANTUM GROUPS

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In the classification of solutions of the Yang–Baxter equation, there are solutions that are not deformations of the trivial solution (essentially the identity). We consider the algebras defined by these solutions, and the corresponding dual algebras. We then study the representations of the latter. We are also interested in the Baxterisation of these R-matrices and in the corresponding quantum planes.

1. Introduction: Five rank-4 *R*-matrices which are not deformations of the identity

We are interested in the study of algebraic structures coming from R-matrices (solutions of the Yang–Baxter equation) that are not deformations of classical ones (i.e., the identity up to signs). Those matrices were obtained by Hlavatý [1] and are also in the classification of Hietarinta [2]. There are five such R-matrices that are invertible. Most of this text is based on the articles [3, 4, 5].

The first cases of interest are

$$R_{H2,3} = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ & 1 & 0 & x_2 \\ & & 1 & x_1 \\ & & & 1 \end{pmatrix} \tag{1}$$

denoted by

- *Exotic 1* (E1) if $x_1 = -x_2 = -h, h_3 \neq -h^2$ [3],
- *Exotic 2* (E2) if $x_1 \neq -x_2$ [3].

Note that if $x_1 = -x_2 = -h, h_3 = -h^2$, then this is $R_{H1,3}$ with $g = -h$ in the notation of Hietarinta, i.e., Jordanian deformation with two parameters.)

The case denoted by *Exotic 3* (E3) is related to the R-matrix

$$R_{S0,2} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ & -1 & 0 & 0 \\ & & -1 & 0 \\ & & & 1 \end{pmatrix} \tag{2}$$

and the algebraic structure was studied in [3].

The cases denoted by “S03” and “S14” come from the R-matrices

$$R_{S0,3} \equiv \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} , \qquad R_{S1,4} \equiv \begin{pmatrix} 0 & 0 & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & 0 & 0 \end{pmatrix} . \tag{3}$$

The structures corresponding to S03 and S14 were studied in [4].

2. Algebra and co-algebra structures

The algebra relations are obtained using

$$R_{12}T_1T_2 = T_2T_1R_{12} \tag{4}$$

$$\text{with} \qquad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad T_1 = T \otimes 1 \qquad T_2 = 1 \otimes T \tag{5}$$

The coalgebra structure is as usual given by $\Delta(T) = T \dot{\otimes} T$
For $R = \mathbb{I}_2 \otimes \mathbb{I}_2$, the relations $RTT = TTR$ are just the commutativity of a, b, c, d ($ab = ba$, $ac = ca$, ...), i.e., T is a matrix of commuting objects. For R of "quantum" type (with two parameters q, p), the relations are

$$ab = qba \qquad ac = pca \qquad bd = pdb \tag{6}$$

$$cd = qdc \qquad qbc = pcb \qquad ad - da = (q - p^{-1})bc \tag{7}$$

which are deformations of simple commutativity relations. The relations we will obtain in the exotic cases are not such deformations.
For the following we use the generators: $\tilde{a} = \frac{1}{2}(a + d)$, $\quad \tilde{d} = \frac{1}{2}(a - d)$.

3. The E1 Case

3.1. *Algebra and coalgebra relations*

The algebra relations (4) in the E1 case are:

$$\tilde{a}c = c\tilde{a} = \tilde{d}c = c\tilde{d} = \tilde{a}\tilde{d} = \tilde{d}\tilde{a} = 0 \ , \quad cb = bc \ , \quad c^2 = 0 \ , \tag{8}$$

$$\tilde{d}b = b\tilde{d} + 2h\tilde{d}^2 + hbc \qquad\qquad \tilde{a}b = b\tilde{a}, \tag{9}$$

and a basis of the enveloping algebra is

$$b^n\tilde{a}^k \ , \quad b^n\tilde{d}^\ell \ , \quad b^n c \ , \qquad n, k \in \mathbb{Z}_+ \ , \quad \ell \in \mathbb{N} \ . \tag{10}$$

h_3 is not explicitly in the relations, but some relations exist uniquely because $h_3 \neq -h_1^2$. This algebra has ideals $I = \mathcal{A}_1 b\tilde{d} \oplus \mathcal{A}_1\tilde{d}^2 \oplus \mathcal{A}_1 bc$, $\quad I_2 = \mathcal{A}_1\tilde{d}^2 \oplus \mathcal{A}_1 bc$ and $I_1 = \mathcal{A}_1 bc$ such that $I_1 \subset I_2 \subset I \subset \mathcal{A}_1$. The dual algebra, or, more precisely, the *algebra in duality*, is defined via non-degenerate pairing $\langle \cdot, \cdot \rangle$, consistent with the co-algebraic structure, i.e. such that,

$$\langle \ u \ , \ ab \ \rangle \ = \ \langle \ \Delta(u) \ , \ a \otimes b \ \rangle \ , \quad \langle \ uv \ , \ a \ \rangle \ = \ \langle \ u \otimes v \ , \ \Delta(a) \ \rangle \tag{11a}$$

$$\langle 1_\mathcal{U}, a \rangle \ = \ \varepsilon_\mathcal{A}(a) \ , \quad \langle u, 1_\mathcal{A} \rangle \ = \ \varepsilon_\mathcal{U}(u) \tag{11b}$$

The algebraic relations are then:

$$[\tilde{D}, C] = -2C \ , \quad [B, C] = \tilde{D} \ , \quad [B, C]_+ = \tilde{D}^2 \tag{12}$$

$$[\tilde{D}, B] = 2B\tilde{D}^2 \ , \quad [\tilde{D}, B]_+ = 0 \ , \quad \tilde{D}^3 = \tilde{D} \ , \quad C^2 = 0 \ , \tag{13}$$

$$[\tilde{A}, Z] = 0 \ , \quad Z = B, C, \tilde{D} \ , \tag{14}$$

$$EZ = ZE = 0 \ , \quad Z = \tilde{A}, B, C, \tilde{D} \tag{15}$$

with co-algebraic structure:

$$\begin{aligned}
&\Delta(\tilde{A}) = \tilde{A}\otimes 1_{\mathcal{U}} + 1_{\mathcal{U}}\otimes\tilde{A}, \quad \Delta(B) = B\otimes 1_{\mathcal{U}} + 1_{\mathcal{U}}\otimes B,\\
&\Delta(C) = C\otimes E + E\otimes C, \quad \Delta(\tilde{D}) = \tilde{D}\otimes E + E\otimes\tilde{D}, \quad \Delta(E) = E\otimes E\\
&\varepsilon_{\mathcal{U}}(Z) = 0\ , \quad Z = \tilde{A}, B, C, \tilde{D}\ , \quad \varepsilon_{\mathcal{U}}(E) = 1
\end{aligned}$$

The extra operator E is defined by: $\langle E, 1_A\rangle = 1$ with all other pairings being zero. Strictly speaking the above algebra is in duality with the factor-algebra since it has zero pairings with the ideals. The full dual is infinitely generated and is under investigation.

Let us make a comparison with FRT duality [7]. We use the relation $\langle\, L^{\pm}\ ,\ T\,\rangle = R^{\pm}$, where: $R^{+} \ \equiv\ P\,R\,P = R(-h)\ ,\quad R^{-}\ \equiv\ R^{-1}$. The comparison leads to

$$L^{\pm}_{11} = L^{\pm}_{22} = e^{-hB}\ , \quad L^{\pm}_{12} = ((h_{\pm} + h^2)B + h\tilde{A})e^{-hB} \tag{16}$$

where $h_{+} = h_3$ and $h_{-} = -h_3 - 2h^2$. We see that $\tilde{A}$ and B are taken into account in this formalism, but that it says nothing about the generators $C, \tilde{D}$.

3.2. *R-matrix minimal polynomials and quantum planes*

In order to address the question of the quantum planes corresponding to the exotic bialgebras we have to know the minimal identity relations which the R-matrices fulfil. As we know the R-matrices producing deformations of the $GL(2)$ and $GL(1|1)$ fulfil second order relations. However, in the cases at hand we have higher order relations. We have:

$$(\hat{R} - \mathrm{id})^2(\hat{R} + \mathrm{id}) = 0, \quad h_1 = -h_2 = h, h_3 \neq -h^2 \tag{17a}$$

$$(\hat{R} - \mathrm{id})^3(\hat{R} + \mathrm{id}) = 0, \quad h_1 + h_2 \neq 0 \tag{17b}$$

$$(\hat{R} - \mathrm{id})(\hat{R} + \mathrm{id}) = 0, \quad h_1 = -h_2 = h, h_3 = -h^2 \tag{17c}$$

where $\hat{R} \equiv PR$, $R = R_{H2,3}$, id is the 4×4 unit matrix. The first case is E1, the second case is E2, the third case is the Jordanian subcase which produces $GL_{h,h}(2)$.

To derive the corresponding quantum planes we shall apply the formalism of [6]. The commutation relations between the coordinates z^i and differentials ζ^i , $(i = 1, 2)$, are given as follows:

$$z^i z^j = \mathcal{P}_{ijk\ell}\, z^k z^\ell\ , \quad \zeta^i\zeta^j = -\mathcal{Q}_{ijk\ell}\,\zeta^k\zeta^\ell\ , \quad z^i\zeta^j = \mathcal{Q}_{ijk\ell}\,\zeta^k z^\ell \tag{18}$$

where the operators $\mathcal{P}$, $\mathcal{Q}$ are functions of $\hat{R}$ and must satisfy:

$$(\mathcal{P} - \mathrm{id})\,(\mathcal{Q} + \mathrm{id}) = 0\ . \tag{19}$$

Thus, there are different choices: four for E1, six for E2 (and just one for $GL_{h,h}(2)$). Choosing $\mathcal{P} - \mathrm{id} = (\hat{R} - \mathrm{id})^a$ with $a = 2, 3, 1$, respectively, and $\mathcal{Q} = \hat{R}$ in all cases. and denoting $(x, y) = (z^1, z^2)$ we obtain

$$xy - yx = hy^2 \ , \qquad h_1 = -h_2 = h \tag{20}$$

for E1 (and the Jordanian), or

$$xy - yx = \frac{1}{2}(h_1 - h_2)y^2 \ , \qquad h_1 \neq -h_2 \tag{21}$$

for E2. We note that the quantum planes corresponding to the three cases are not essentially different.

Denoting $(\xi, \eta) = (\zeta^1, \zeta^2)$ we obtain

$$\xi^2 + \frac{h_1 - h_2}{2}\, \xi\eta = 0 \ , \quad \eta^2 = 0 \ , \quad \xi\eta = -\eta\xi \tag{22}$$

for E2, while for E1 $\xi^2 + h\,\xi\eta = 0$, which is valid also for the Jordanian subcase.

Finally, for the coordinates-differentials relations we obtain

$$x\xi = \xi x + h_1\xi y + h_2\eta x + h_3\eta y \ , \quad x\eta = \eta x + h_1\eta y \ , \tag{23}$$

$$y\xi = \xi y + h_2\eta y \ , \qquad\qquad y\eta = \eta y \tag{24}$$

4. The S03 case

4.1. *Algebraic relations*

The algebra relations in the S03 case are:

$$\tilde{b}^2 = \tilde{c}^2 = 0 \ , \qquad \tilde{a}\tilde{d} = \tilde{d}\tilde{a} = 0 \ , \qquad \tilde{a}\tilde{b} = 0 \ , \tag{25}$$

$$\tilde{b}\tilde{d} = 0 \ , \qquad \tilde{d}\tilde{c} = 0 \ , \qquad \tilde{c}\tilde{a} = 0 \ . \tag{26}$$

where: $\tilde{b} = \frac{1}{2}(b + c)$, $\tilde{c} = \frac{1}{2}(b - c)$.

There is no PBW basis in this case. Indeed, the ordering is cyclic:

$$\tilde{a} \ > \ \tilde{c} \ > \ \tilde{d} \ > \ \tilde{b} \ > \ \tilde{a} \tag{27}$$

Thus, the basis consists of building blocks like $\tilde{a}^k\, \tilde{c}\, \tilde{d}^\ell\, \tilde{b}$ and cyclic. Explicitly the basis can be described by the following monomials:

$$\tilde{a}^{k_1}\, \tilde{c}\, \tilde{d}^{\ell_1}\, \tilde{b} \cdots \tilde{a}^{k_n}\, \tilde{c}\, \tilde{d}^{\ell_n}\, \tilde{b}\, \tilde{a}^{k_{n+1}} \ , \qquad \tilde{d}^{\ell_1}\, \tilde{b}\, \tilde{a}^{k_1}\, \tilde{c} \cdots \tilde{d}^{\ell_n}\, \tilde{b}\, \tilde{a}^{k_n} \ ,$$

$$\tilde{a}^{k_1}\, \tilde{c}\, \tilde{d}^{\ell_1}\, \tilde{b} \cdots \tilde{a}^{k_n}\, \tilde{c}\, \tilde{d}^{\ell_n} \ , \qquad \tilde{d}^{\ell_1}\, \tilde{b}\, \tilde{a}^{k_1}\, \tilde{c} \cdots \tilde{d}^{\ell_n}\, \tilde{b}\, \tilde{a}^{k_n}\, \tilde{c}\, \tilde{d}^{\ell_{n+1}} \ ,$$

where in all cases $n\,,k_i\,,\ell_i\,\in\,\mathbb{Z}_+$.
The algebra in duality is given by

$$\begin{aligned}
&[\tilde{A},Z]=0\ ,\quad Z=\tilde{B},\tilde{C}\ ,\quad \tilde{A}\tilde{D}=\tilde{D}\tilde{A}=\tilde{D}^3=\tilde{B}^2\tilde{D}=\tilde{D}\tilde{B}^2=\tilde{D}\ ,\\
&[\tilde{B},\tilde{C}]=-2\tilde{D}\ ,\quad \tilde{D}\tilde{B}=-\tilde{B}\tilde{D}=\tilde{C}\tilde{D}^2=\tilde{D}^2\tilde{C}\ ,\quad \{\tilde{C},\tilde{D}\}=0\ ,\\
&\tilde{B}^2+\tilde{C}^2=0\ ,\quad \tilde{B}^3=\tilde{B}\ ,\quad \tilde{C}^3=-\tilde{C}\ ,\quad \tilde{B}^2\tilde{A}=\tilde{A}\ ,
\end{aligned}\tag{29}$$

with coproduct:

$$\Delta_{\mathcal{U}}(\tilde{A})=\tilde{A}\otimes 1_{\mathcal{U}}+1_{\mathcal{U}}\otimes\tilde{A}\tag{30}$$
$$\Delta_{\mathcal{U}}(\tilde{B})=\tilde{B}\otimes 1_{\mathcal{U}}+(1_{\mathcal{U}}-\tilde{B}^2)\otimes\tilde{B}\tag{31}$$
$$\Delta_{\mathcal{U}}(\tilde{C})=\tilde{C}\otimes(1_{\mathcal{U}}-\tilde{B}^2)+1_{\mathcal{U}}\otimes\tilde{C}\tag{32}$$
$$\Delta_{\mathcal{U}}(\tilde{D})=\tilde{D}\otimes(1_{\mathcal{U}}-\tilde{B}^2)+(1_{\mathcal{U}}-\tilde{B}^2)\otimes\tilde{D}\tag{33}$$
$$\varepsilon_{\mathcal{U}}(Z)=0\ ,\qquad Z=\tilde{A},\tilde{B},\tilde{C},\tilde{D}\ .\tag{34}$$

$\tilde{A}$, $\tilde{B}^2=-\tilde{C}^2$ and $\tilde{D}^2$ are Casimir operators.
The bialgebra $s03$ is not a Hopf algebra (since there is no antipode).

The algebra generated by the generator $\tilde{A}$ is a sub-bialgebra of $s03$. The algebra $s03'$ generated by $\tilde{B},\tilde{C},\tilde{D}$ is a nine-dimensional sub-bialgebra of $s03$ with PBW basis:

$$1_{\mathcal{U}}\,,\ \tilde{B},\ \tilde{C},\ \tilde{D},\ \tilde{B}\tilde{C},\ \tilde{B}\tilde{D},\ \tilde{D}\tilde{C},\ \tilde{B}^2,\ \tilde{D}^2\tag{35}$$

The algebra $s03$ is not the direct sum of the two subalgebras described above since both subalgebras have nontrivial action on each other, e.g., $\tilde{B}^2\tilde{A}=\tilde{A}$, $\tilde{A}\tilde{D}=\tilde{D}$. The algebra $s03$ is a nine-dimensional associative algebra over the central algebra generated by $\tilde{A}$.

Let us again make a comparison with FRT duality.
$L^{\pm}$ are matrices of operators $L^{\pm}_{ij}$ $(i,j=1,2)$ satisfying the relations

$$R^+L^+_1L^+_2=L^+_2L^+_1R^+\tag{36}$$
$$R^+L^-_1L^-_2=L^-_2L^-_1R^+\tag{37}$$
$$R^+L^+_1L^-_2=L^-_2L^+_1R^+\tag{38}$$

with $L_1\equiv L\otimes 1$, $L_2\equiv 1\otimes L$. Explicitly, these RLL relations read

$$\begin{aligned}
&(L^{\pm}_{11})^2=(L^{\pm}_{22})^2 &&[L^{\pm}_{11},L^{\pm}_{22}]=0\\
&(L^{\pm}_{12})^2=-(L^{\pm}_{21})^2 &&[L^{\pm}_{12},L^{\pm}_{21}]_+=0\\
&L^{\pm}_{11}L^{\pm}_{12}=L^{\pm}_{22}L^{\pm}_{21} &&L^{\pm}_{11}L^{\pm}_{21}=L^{\pm}_{22}L^{\pm}_{12}\\
&L^{\pm}_{12}L^{\pm}_{11}=-L^{\pm}_{21}L^{\pm}_{22} &&L^{\pm}_{12}L^{\pm}_{22}=-L^{\pm}_{21}L^{\pm}_{11}
\end{aligned}\tag{39}$$

and for the RL^+L^- ones

$$L^+_{ij}L^-_{kl} - L^-_{ij}L^+_{kl} + \theta_i L^+_{\bar{i}j}L^-_{\bar{k}l} + \theta_j L^-_{i\bar{j}}L^+_{k\bar{l}} = 0 \tag{40}$$

with $\bar{n} \equiv 3-n$, $\theta_1 = 1$, $\theta_2 = -1$. In the case of $s03$, the FRT relations in the dual algebra are richer than the relations given using only (11). They will indeed lead to more irreducible finite dimensional representations.
A convenient basis is given by

$$\begin{aligned} &\tilde{L}^\pm_{11} = L^\pm_{11} + L^\pm_{22} && \tilde{L}^\pm_{22} = L^\pm_{11} - L^\pm_{22} \\ &\tilde{L}^\pm_{12} = L^\pm_{12} + L^\pm_{21} && \tilde{L}^\pm_{21} = L^\pm_{12} - L^\pm_{21} \end{aligned} \tag{41}$$

In this basis, the relations (39) read

$$\begin{aligned} &\tilde{L}^\pm_{11}\,\tilde{L}^\pm_{22} = 0 && \tilde{L}^\pm_{22}\,\tilde{L}^\pm_{11} = 0 \\ &(\tilde{L}^\pm_{12})^2 = 0 && (\tilde{L}^\pm_{21})^2 = 0 \\ &\tilde{L}^\pm_{11}\,\tilde{L}^\pm_{21} = 0 && \tilde{L}^\pm_{12}\,\tilde{L}^\pm_{11} = 0 \\ &\tilde{L}^\pm_{21}\,\tilde{L}^\pm_{22} = 0 && \tilde{L}^\pm_{22}\,\tilde{L}^\pm_{12} = 0 \end{aligned} \tag{42}$$

whereas the relations (40) become

$$\begin{aligned} &[\tilde{L}^+_{11}, \tilde{L}^-_{11}] = 0 && \tilde{L}^-_{21}\,\tilde{L}^+_{11} = \tilde{L}^+_{21}\,\tilde{L}^-_{11} \\ &\tilde{L}^-_{11}\,\tilde{L}^+_{12} = \tilde{L}^+_{11}\,\tilde{L}^-_{12} && \tilde{L}^-_{21}\,\tilde{L}^+_{12} = \tilde{L}^+_{21}\,\tilde{L}^-_{12} \\ &\tilde{L}^-_{11}\,\tilde{L}^+_{21} = \tilde{L}^+_{21}\,\tilde{L}^-_{22} && \tilde{L}^-_{21}\,\tilde{L}^+_{21} = -\tilde{L}^+_{11}\,\tilde{L}^-_{22} \\ &\tilde{L}^-_{11}\,\tilde{L}^+_{22} = \tilde{L}^+_{21}\,\tilde{L}^-_{21} && \tilde{L}^-_{21}\,\tilde{L}^+_{22} = -\tilde{L}^+_{11}\,\tilde{L}^-_{21} \\ \\ &\tilde{L}^-_{12}\,\tilde{L}^+_{11} = -\tilde{L}^+_{22}\,\tilde{L}^-_{12} && \tilde{L}^-_{22}\,\tilde{L}^+_{11} = \tilde{L}^+_{12}\,\tilde{L}^-_{12} \\ &\tilde{L}^-_{12}\,\tilde{L}^+_{12} = -\tilde{L}^+_{22}\,\tilde{L}^-_{11} && \tilde{L}^-_{22}\,\tilde{L}^+_{12} = \tilde{L}^+_{12}\,\tilde{L}^-_{11} \\ &\tilde{L}^-_{12}\,\tilde{L}^+_{21} = \tilde{L}^+_{12}\,\tilde{L}^-_{21} && \tilde{L}^-_{22}\,\tilde{L}^+_{21} = \tilde{L}^+_{22}\,\tilde{L}^-_{21} \\ &\tilde{L}^-_{12}\,\tilde{L}^+_{22} = \tilde{L}^+_{12}\,\tilde{L}^-_{22} && [\tilde{L}^+_{22}, \tilde{L}^-_{22}] = 0 \end{aligned} \tag{43}$$

Denote for $n \geq 1$:

$$F_n(k_i; l_i) \equiv \prod_{i=1}^{n} \tilde{L}^{+k_i}_{11}\,\tilde{L}^+_{12}\tilde{L}^{+l_i}_{22}\,\tilde{L}^+_{21} \tag{44}$$

$$G_n(l_i; k_i) \equiv \prod_{i=1}^{n} \tilde{L}^{+l_i}_{22}\,\tilde{L}^+_{21}\tilde{L}^{+k_i}_{11}\,\tilde{L}^+_{12} \tag{45}$$

and for $n = 0$

$$F_0(k_i; l_i) \equiv 1; \quad G_0(l_i; k_i) \equiv 1 \tag{46}$$

The basis elements of the algebra generated by the $\tilde{L}^+$'s are (following [4])

$$\begin{aligned} &F_n(k_i;l_i)\tilde{L}_{11}^{+k_{n+1}}; \quad F_{n-1}(k_i;l_i)\tilde{L}_{11}^{+k_n}\tilde{L}_{12}^{+}\tilde{L}_{22}^{+l_n}; \\ &G_n(l_i;k_i)\tilde{L}_{22}^{+l_{n+1}}; \quad G_{n-1}(l_i;k_i)\tilde{L}_{22}^{+l_n}\tilde{L}_{21}^{+}\tilde{L}_{11}^{+k_n} \end{aligned} \tag{47}$$

Defining also $K_n = \sum_{i=1}^n k_i, \quad L_n = \sum_{i=1}^n l_i$ the actions of generators $\tilde{L}^-$ on the basis elements are, e.g.,

$$\begin{aligned} \tilde{L}_{11}^- F_n(k_i;l_i) &= F_{n-1}(k_1+1,k_i;l_i)\tilde{L}_{11}^{+k_n}\tilde{L}_{12}^{+}\tilde{L}_{22}^{+l_n}\tilde{L}_{21}^- \\ \tilde{L}_{12}^- F_n(k_i;l_i) &= (-1)^{K_n+L_n+1}G_{n-1}(k_1+1,k_i;l_i)\tilde{L}_{22}^{+k_n}\tilde{L}_{21}^{+}\tilde{L}_{11}^{+l_n}\tilde{l}_{22}^- \\ \tilde{L}_{21}^- F_n(k_i;l_i) &= G_n(0,..,l_{n-1};k_i)\tilde{L}_{22}^{+l_n}\tilde{L}_{21}^- \\ \tilde{L}_{22}^- F_n(k_i,;l_i) &= (-1)^{K_n+L_n}F_n(0,..,l_{n-1};k_i)\tilde{L}_{11}^{+l_n}\tilde{L}_{22}^- \end{aligned} \tag{48}$$

$$\begin{aligned} \tilde{L}_{11}^- G_n(l_i;k_i) &= (-1)^{K_n+L_n}G_n(0,..,k_{n-1};l_i)\tilde{L}_{22}^{+k_n}\tilde{L}_{11}^- \\ \tilde{L}_{12}^- G_n(l_i;k_i) &= F_n(0,..,k_{n-1};l_i)\tilde{L}_{11}^{+k_n}\tilde{l}_{12}^- \end{aligned} \tag{49}$$

$$\begin{aligned} \tilde{L}_{21}^- G_n(l_i;k_i) &= (-1)^{K_n+L_n+1}F_{n-1}(l_1+1,l_i;k_i)\tilde{L}_{11}^{+l_n}\tilde{L}_{12}^{+}\tilde{L}_{22}^{+k_n}\tilde{L}_{11}^- \\ \tilde{L}_{22}^- G_n(l_i;k_i) &= G_{n-1}(l_1+1,l_i;k_i)\tilde{L}_{22}^{+l_n}\tilde{L}_{21}^{+}\tilde{L}_{11}^{+k_n}\tilde{L}_{12}^- \end{aligned} \tag{50}$$

These equations allow one to order the $\tilde{L}^-$ with respect to the $\tilde{L}^+$. For the $\tilde{L}^-$ among themselves, there exists a basis similar to (47).

4.2. *Finite dimensional irreducible representations*

Let us first consider the algebra generated by A, B, C, D and the relations (29). Since $\tilde{D}^3 = \tilde{D}$, there exists a weight vector v_0 such that: $\tilde{D}\, v_0 = \lambda\, v_0$ where $\lambda^3 = \lambda$. The finite dimensional irreps are then

- one-dimensional trivial
- two-dimensional with Casimir values $\mu, 1, 1$ for $\tilde{A}, \tilde{B}^2, \tilde{D}^2$, respectively, $\mu \in \mathbb{C}$.
- one-dimensional with Casimir values $\mu, 1, 0$ for $\tilde{A}, \tilde{B}^2, \tilde{D}^2$, respectively, $\mu \in \mathbb{C}$.

A two-dimensional representation for the algebra generated by the L-operators is provided by the R-matrix itself, setting $\pi(L^+) = R_{21}$, $\pi(L^-) = R^{-1}$ (see [7, 8])

$$\begin{aligned} \pi(L_{11}^{\pm}) &= \begin{pmatrix} 1 & 0 \\ 0 & \mp 1 \end{pmatrix} \qquad & \pi(L_{12}^{\pm}) &= \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix} \\ \pi(L_{21}^{\pm}) &= \begin{pmatrix} 0 & 1 \\ \mp 1 & 0 \end{pmatrix} \qquad & \pi(L_{22}^{\pm}) &= \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \tag{51}$$

Note that this does not exhaust the set of two-dimensional representations.

Let N_1 and N_2 be two non negative integers. Here is an example (in the $\tilde{L}$ basis) of a finite dimensional irreducible representation of arbitrary dimension $N_1 + N_2$.

$$\pi(\tilde{L}_{11}) = \mathrm{diag}(\rho_1, \cdots, \rho_{N_1}, \underbrace{0, \cdots, 0}_{N_2}) \qquad \rho_i \neq \rho_j \quad \text{for} \quad i \neq j \tag{52}$$

$$\pi(\tilde{L}_{22}) = \mathrm{diag}(\underbrace{0, \cdots, 0}_{N_1}, \lambda_1, \cdots, \lambda_{N_2}) \qquad \lambda_i \neq \lambda_j \quad \text{for} \quad i \neq j \tag{53}$$

$$\Big(\pi(\tilde{L}_{12})\Big)_{ij} \neq 0 \quad \text{iff} \quad i \in \{1, \cdots, N_1\}, \quad j \in \{N_1 + 1, \cdots, N_1 + N_2\} \tag{54}$$

$$\Big(\pi(\tilde{L}_{21})\Big)_{ij} \neq 0 \quad \text{iff} \quad i \in \{N_1 + 1, \cdots, N_1 + N_2\}, \quad j \in \{1, \cdots, N_1\} \tag{55}$$

4.3. *Baxterisation*

We introduce the following Ansatz (choosing a convenient normalisation):

$$\hat{R}(x) = I + c(x)\hat{R} \tag{56}$$

and we try to find $c(x)$ such that $\hat{R}(x)$ would satisfy the parametrised Yang–Baxter equation:

$$\hat{R}_{(12)}(x)\hat{R}_{(23)}(xy)\hat{R}_{(12)}(y) = \hat{R}_{(23)}(y)\hat{R}_{(12)}(xy)\hat{R}_{(23)}(x) \tag{57}$$

The result is:

$$\hat{R}(x) = (\sqrt{2}x)^{-1}\hat{R} + (\sqrt{2}x)\hat{R}^{-1} = \frac{1}{\sqrt{2x}} \begin{pmatrix} x+1 & 0 & 0 & 1-x \\ 0 & x+1 & x-1 & 0 \\ 0 & 1-x & x+1 & 0 \\ x-1 & 0 & 0 & x+1 \end{pmatrix} \tag{58}$$

4.4. *Spectral decomposition and noncommutative planes*

The minimal polynomial identity satisfied by $\mathcal{R}$ is

$$\hat{R}^2 - 2\hat{R} + 2I = 0 \tag{59}$$

and we have the spectral decomposition:

$$\hat{R} = (1-i)P_{(+)} + (1+i)P_{(-)} = (1+i)I - 2iP_{(+)} \tag{60}$$

$$\text{where} \qquad P_{(\pm)} \;\equiv\; \frac{1}{2}(I \pm i(\hat{R} - I)) \tag{61}$$

are projectors resolving the identity: $P_{(i)}P_{(j)} = \delta_{ij}P_{(i)}$, $i, j = \pm$, $P_{(+)} + P_{(-)} = I$. The quantum plane relations in the case of S03 are then

$$x_1{}^2 = x_1x_2, \quad x_2{}^2 = -x_2x_1 \ , \quad \xi_1{}^2 = -\xi_1\xi_2, \quad \xi_2{}^2 = \xi_2\xi_1 \tag{62}$$

$$x_1\xi_1 = (\nu-1)\xi_1x_1 + \nu\xi_1x_2 \ , \quad x_1\xi_2 = (\nu-1)\xi_1x_2 + \nu\xi_1x_1 \tag{63}$$

$$x_2\xi_1 = (\nu-1)\xi_2x_1 - \nu\xi_2x_2 \ , \quad x_2\xi_2 = (\nu-1)\xi_2x_2 - \nu\xi_2x_1 \tag{64}$$

where ν is arbitrary real parameter.

5. The S14 case

The relations in the S14 case are:

$$\tilde{b}\tilde{c} + \tilde{c}\tilde{b} = 0 \qquad\qquad \tilde{a}\tilde{d} + \tilde{d}\tilde{a} = 0$$
$$\tilde{a}\tilde{b} = \tilde{b}\tilde{a} = \tilde{a}\tilde{c} = \tilde{c}\tilde{a} = \tilde{b}\tilde{d} = \tilde{d}\tilde{b} = \tilde{c}\tilde{d} = \tilde{d}\tilde{c} = 0$$

and the dual algebra is:

$$\tilde{C} = \tilde{D}\tilde{B} = -\tilde{B}\tilde{D} \ , \quad [\tilde{A}, \tilde{D}] = 0 \ , \quad EZ = ZE = 0 \tag{65a}$$

$$\tilde{A}\tilde{B} = \tilde{B}\tilde{A} = \tilde{D}^2\tilde{B} = \tilde{B}^3 = \tilde{B} \ , \quad Z = \tilde{A}, \tilde{B}, \tilde{D} \ . \tag{65b}$$

The dual coalgebra is given by (with $K \equiv (-1)^{\tilde{A}}$)

$$\Delta_{\mathcal{U}}(\tilde{A}) = \tilde{A} \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes \tilde{A} \ , \quad \Delta_{\mathcal{U}}(\tilde{B}) = \tilde{B} \otimes E + E \otimes \tilde{B} \tag{66a}$$

$$\Delta_{\mathcal{U}}(\tilde{D}) = \tilde{D} \otimes K + 1_{\mathcal{U}} \otimes \tilde{D} \ , \qquad \Delta(E) = E \otimes E \tag{66b}$$

$$\varepsilon_{\mathcal{U}}(Z) = 0 \ \text{ for } \ Z = \tilde{A}, \tilde{B}, \tilde{D} \ ; \qquad \varepsilon_{\mathcal{U}}(E) = 1 \tag{66c}$$

The irreducible representations of $s14$ follow the classification

- one-dimensional with Casimir values $\mu, 0, \lambda^2$ for $\tilde{A}, \tilde{B}^2, \tilde{D}^2$, respectively, $\mu, \lambda \in \mathbb{C}$.
- two two-dimensional with all Casimirs $\tilde{A}, \tilde{B}^2, \tilde{D}^2$ having the value 1.

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